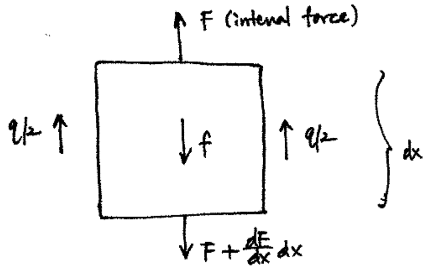
(a)

FBD. inside the pile



Equilibrium:

$$\Sigma F_x = 0 \Leftrightarrow \left(F + \frac{dF}{dx}dx\right) - F + f\,dx - q\,dx = 0$$

$$\overset{F=A\sigma}{\Leftrightarrow} \quad \frac{d}{dx}(A\sigma) + f - q = 0$$

Constitutive equations:

$$\sigma = E\varepsilon, \quad q = ku.$$

Kinematics:

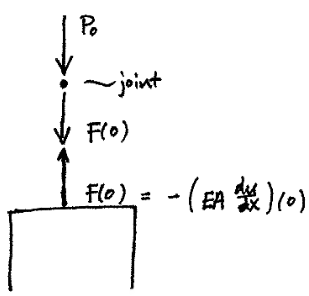
$$\varepsilon = \frac{du}{dx}.$$

Combine these equations to obtain the governing equation in terms of the displacement u:

$$-\frac{d}{dx}\left(EA\frac{du}{dx}\right) + ku = f, \quad \forall x \in (0, L).$$

What are the boundary conditions? The BC at $x=0$ can be determined from considering the FBD of the "joint."

FBD. of the joint at $x=0$.



Equilibrium $\Leftrightarrow -F(0) + P_0 = 0$

$$\Leftrightarrow -\left(EA\frac{du}{dx}\right)(0) = P_0$$

And, of course,

$$u(L) = 0.$$

Strong form.

Find $u \in C^2(0, L)$ such that

$$\begin{cases} -\dfrac{d}{dx}\left(EA\dfrac{du}{dx}\right) + ku = f, & \forall x \in (0, L). \\ -\left(EA\dfrac{du}{dx}\right)(0) = P_0 \\ u(L) = 0 \end{cases}$$

(b)

Multiply the governing equation by an admissible test function φ, integrate from 0 to L, and do IBP. Then,

$$\int_0^L \left[-\frac{d}{dx}\left(EA\frac{du}{dx}\right) + ku \right] \varphi \, dx = \int_0^L f\varphi \, dx$$

$$\Rightarrow \int_0^L \left[EA \frac{du}{dx}\frac{d\varphi}{dx} + ku\varphi \right] dx - \left[EA \frac{du}{dx} \varphi \right]_0^L = \int_0^L f\varphi \, dx$$

$$\Rightarrow \int_0^L \left[EA \frac{du}{dx}\frac{d\varphi}{dx} + ku\varphi \right] dx = \int_0^L f\varphi \, dx + \left(EA\frac{du}{dx}\right)(L)\underbrace{\varphi(L)}_{\equiv 0} \underbrace{- \left(EA\frac{du}{dx}\right)(0)}_{= P_0}\varphi(0)$$

$$\Rightarrow \int_0^L \left[\underbrace{EA \frac{du}{dx}\frac{d\varphi}{dx}}_{\text{internal virtual work in the pile}} + \underbrace{ku\varphi}_{\text{IVW in the soil}} \right] dx = \underbrace{\int_0^L f\varphi \, dx + P_0\varphi(0)}_{\text{external virtual work due to the distributed load and tip force}}$$

Weak form.

Find $u \in H^1(0,L)$ with $u(L) = 0$, such that

$$\int_0^L \left[EA \frac{du}{dx}\frac{d\varphi}{dx} + ku\varphi \right] dx = \int_0^L f\varphi \, dx + P_0\varphi(0),$$

for all $\varphi \in H^1(0,L)$ with $\varphi(L) = 0$.

· Note the difference between the Dirichlet BC $u(L) = 0$ (we require explicitly that the solution satisfies this – "essential"!) and the Neumann BC $-\left(EA\frac{du}{dx}\right)(0) = P_0$ (we embedded this in the weak form – "natural"!)

Finite-dimensional subspace

Consider an element with $(p+1)$ basis functions. Let,

$$N^e = [N_1 \;\; N_2 \;\; \cdots \;\; N_{p+1}]$$

$$B^e = [N_1' \;\; N_2' \;\; \cdots \;\; N_{p+1}']$$

Then,

$$(K^e)_{ij} = \int_e \left[EA \frac{dN_j}{dx}\frac{dN_i}{dx} + k N_j N_i \right] dx \quad \Rightarrow \quad K^e = \int_e EA (B^e)^T(B^e) + k(N^e)^T(N^e) \, dx$$

$$(f^e)_i = \int_e f N_i \, dx + P_0 \underbrace{N_i(0)}_{= \begin{cases} 1, & \text{if } e=1 \text{ and } i=1 \\ 0, & \text{otherwise} \end{cases}} \quad \Rightarrow \quad f^e = \int_e f (N^e)^T dx + \begin{cases} \begin{bmatrix} P_0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, & e=1, i=1 \\ \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, & \text{otherwise.} \end{cases}$$

(b1). For all $\varphi \in H^1(0,L)$ with $\varphi(L)=0$. We have that,

$$\int_0^L \left[EA \frac{du}{dx}\frac{d\varphi}{dx} + ku\varphi\right] dx = \int_0^L f\varphi \, dx + P_0\varphi(0)$$

$$\Rightarrow \int_0^L \left[-\frac{d}{dx}\left(EA\frac{du}{dx}\right)\varphi + ku\varphi\right] dx + \underbrace{\left[EA\frac{du}{dx}\varphi\right]_0^L}_{=0 \text{ at } x=L} = \int_0^L f\varphi \, dx + P_0\varphi(0).$$

$$\Rightarrow \boxed{\int_0^L \left[-\frac{d}{dx}\left(EA\frac{du}{dx}\right) + ku - f\right]\varphi \, dx = \left(EA\frac{du}{dx}(0) + P_0\right)\varphi(0).} \qquad \text{(Eq. 1)}$$

We choose test functions of the form,

$$\varphi = \left[-\frac{d}{dx}\left(EA\frac{du}{dx}\right) + ku - f\right]\Phi \qquad (\Phi \text{ is a Cauchy's infinitely differentiable function})$$

where,

Φ (1) is $C^\infty(0,L)$, infinitely smooth.

(2) satisfies homogeneous Dirichlet BCs $\Phi(0)=0$, $\Phi(L)=0$,

(3) is positive in the interval $(0,L)$. ie. $\Phi(x)>0$ for all $x\in(0,L)$.

Then,

$$\int_0^L \underbrace{\left[-\frac{d}{dx}\left(EA\frac{du}{dx}\right) + ku - f\right]^2}_{\geq 0} \underbrace{\Phi}_{>0} \, dx = 0 \quad , \text{ for all such } \Phi\text{'s}.$$

$$\Rightarrow \boxed{-\frac{d}{dx}\left(EA\frac{du}{dx}\right) + ku - f = 0, \quad \text{for all } x\in(0,L)}$$

Go back to (Eq. 1), and consider all test functions $\varphi\in H^1(0,L)$ with $\varphi(L)=0$. Since there is no restriction on the value of $\varphi(0)$, we must have that

$$\boxed{EA\frac{du}{dx}(0) + P_0 = 0}.$$

Of course, the other BC $u(L)=0$ still remains in effect.

(c)

For $e=1$,

$$N^e = \left[1-\frac{x}{L}, \ \frac{x}{L} \right]$$

$$B^e = \left[-\frac{1}{L}, \ \frac{1}{L} \right].$$

$$K^e = \int_0^L EA \begin{bmatrix} (-\frac{1}{L})^2 & (-\frac{1}{L})(\frac{1}{L}) \\ \text{sym.} & (\frac{1}{L})^2 \end{bmatrix} + \frac{EA}{L^2} \begin{bmatrix} (1-\frac{x}{L})^2 & (1-\frac{x}{L})(\frac{x}{L}) \\ \text{sym.} & (\frac{x}{L})^2 \end{bmatrix} dx$$

$$= EA \begin{bmatrix} \frac{1}{L} & -\frac{1}{L} \\ -\frac{1}{L} & \frac{1}{L} \end{bmatrix} + \frac{EA}{L^2} \begin{bmatrix} \frac{L}{3} & \frac{L}{6} \\ \frac{L}{6} & \frac{L}{3} \end{bmatrix}$$

$$= \frac{EA}{L} \begin{bmatrix} \frac{4}{3} & -\frac{5}{6} \\ -\frac{5}{6} & \frac{4}{3} \end{bmatrix}$$

$$f^e = \int_0^L \frac{P_0}{L} \begin{bmatrix} 1-\frac{x}{L} \\ \frac{x}{L} \end{bmatrix} dx + P_0 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \frac{P_0}{L} \begin{bmatrix} \frac{L}{2} \\ \frac{L}{2} \end{bmatrix} + P_0 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= P_0 \begin{bmatrix} \frac{3}{2} \\ \frac{1}{2} \end{bmatrix}.$$

Assemble.

$$\frac{EA}{L} \begin{bmatrix} \frac{4}{3} & -\frac{5}{6} \\ -\frac{5}{6} & \frac{4}{3} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = P_0 \begin{bmatrix} \frac{3}{2} \\ \frac{1}{2} \end{bmatrix}.$$

and apply the BC, $u_2 = 0$. Then,

$$u_1 = \frac{9}{8} \frac{P_0 L}{EA}, \qquad \boxed{u_h(x) = \frac{9}{8} \frac{P_0 L}{EA} \cdot \left(1-\frac{x}{L}\right) + 0 \cdot \left(\frac{x}{L}\right). \quad \text{for } 0 \le x \le L}.$$

Force at the pile head:

$$F_h(0) = -\left(EA \frac{du_h}{dx}\right)(0)$$

$$= \frac{9}{8} P_0 . \qquad (\text{recall that } F(0) = P_0).$$

The internal force at the pile head is <u>approximate</u>, because we did not explicitly enforce the condition $F(0) = P_0$ in the weak form.

(d) For $e=1$,

$$N^e = \left[\, 1 - \frac{x}{L/2} \,,\quad \frac{x}{L/2} \,\right]$$

$$B^e = \left[\, -\frac{1}{L/2} \,,\quad \frac{1}{L/2} \,\right].$$

$$K^e = \int_0^{L/2} EA \begin{bmatrix} \left(-\frac{1}{L/2}\right)^2 & \left(-\frac{1}{L/2}\right)\left(\frac{1}{L/2}\right) \\ \text{sym.} & \left(\frac{1}{L/2}\right)^2 \end{bmatrix} + \frac{EA}{L^2} \begin{bmatrix} \left(1-\frac{x}{L/2}\right)^2 & \left(1-\frac{x}{L/2}\right)\left(\frac{x}{L/2}\right) \\ \text{sym.} & \left(\frac{x}{L/2}\right)^2 \end{bmatrix} dx$$

$$= EA \begin{bmatrix} \frac{2}{L} & -\frac{2}{L} \\ -\frac{2}{L} & \frac{2}{L} \end{bmatrix} + \frac{EA}{L^2} \begin{bmatrix} \frac{L}{6} & \frac{L}{12} \\ \frac{L}{12} & \frac{L}{6} \end{bmatrix}$$

$$= \frac{EA}{L} \begin{bmatrix} \frac{13}{6} & -\frac{23}{12} \\ -\frac{23}{12} & \frac{13}{6} \end{bmatrix}$$

$$f^e = \int_0^{L/2} \frac{P_0}{L} \begin{bmatrix} 1 - \frac{x}{L/2} \\ \frac{x}{L/2} \end{bmatrix} dx + P_0 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \frac{P_0}{L} \begin{bmatrix} \frac{L}{4} \\ \frac{L}{4} \end{bmatrix} + P_0 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= P_0 \begin{bmatrix} \frac{5}{4} \\ \frac{1}{4} \end{bmatrix}$$

For $e=2$,

$$N^e = \left[\, 1 - \frac{x-\frac{L}{2}}{L/2} \,,\quad \frac{x-\frac{L}{2}}{L/2} \,\right]$$

$$B^e = \left[\, -\frac{1}{L/2} \,,\quad \frac{1}{L/2} \,\right]$$

$$K^e = \int_{L/2}^{L} EA \begin{bmatrix} \left(-\frac{1}{L/2}\right)^2 & \left(-\frac{1}{L/2}\right)\left(\frac{1}{L/2}\right) \\ \text{sym.} & \left(\frac{1}{L/2}\right)^2 \end{bmatrix} + \frac{EA}{L^2} \begin{bmatrix} \left(1-\frac{2}{L}\left(x-\frac{L}{2}\right)\right)^2 & \left(1-\frac{2}{L}\left(x-\frac{L}{2}\right)\right)\left(\frac{2}{L}\left(x-\frac{L}{2}\right)\right) \\ \text{sym.} & \left(\frac{2}{L}\left(x-\frac{L}{2}\right)\right)^2 \end{bmatrix} dx$$

$$= EA \begin{bmatrix} \frac{2}{L} & -\frac{2}{L} \\ -\frac{2}{L} & \frac{2}{L} \end{bmatrix} + \frac{EA}{L^2} \begin{bmatrix} \frac{L}{6} & \frac{L}{12} \\ \frac{L}{12} & \frac{L}{6} \end{bmatrix}$$

$$f^e = \int_{L/2}^{L} \frac{P_0}{L} \begin{bmatrix} 1 - \frac{2}{L}\left(x-\frac{L}{2}\right) \\ \frac{2}{L}\left(x-\frac{L}{2}\right) \end{bmatrix} dx + P_0 \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$= \frac{P_0}{L} \begin{bmatrix} \frac{L}{4} \\ \frac{L}{4} \end{bmatrix} + P_0 \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$= P_0 \begin{bmatrix} \frac{1}{4} \\ \frac{1}{4} \end{bmatrix}$$

Assemble,

$$\frac{EA}{L}\begin{bmatrix} \frac{13}{6} & -\frac{23}{12} & \\ -\frac{23}{12} & \frac{13}{6}+\frac{13}{6} & -\frac{23}{12} \\ & -\frac{23}{12} & \frac{13}{6} \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = P_0\begin{bmatrix} \frac{5}{4} \\ \frac{1}{4}+\frac{1}{4} \\ \frac{1}{4} \end{bmatrix}$$

Apply the BC $u_3 = 0$, and solve for u_1, u_2:

$$\frac{EA}{L}\begin{bmatrix} \frac{13}{6} & -\frac{23}{12} \\ -\frac{23}{12} & \frac{13}{3} \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = P_0\begin{bmatrix} \frac{5}{4} \\ \frac{1}{2} \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \frac{P_0 L}{EA}\begin{bmatrix} \frac{918}{823} \\ \frac{501}{823} \end{bmatrix} \approx \frac{P_0 L}{EA}\begin{bmatrix} 1.11543 \\ 0.60875 \end{bmatrix}.$$

Hence,

$$u_h(x) \approx \begin{cases} 1.11543\,\frac{P_0 L}{EA}\left(1-\frac{2}{L}x\right) + 0.60875\,\frac{P_0 L}{EA}\left(\frac{2}{L}x\right), & \text{for } 0 \le x \le \frac{L}{2} \\ 0.60875\,\frac{P_0 L}{EA}\left(1-\frac{2}{L}\left(x-\frac{L}{2}\right)\right) + 0\,\frac{P_0 L}{EA}\left(\frac{2}{L}\left(x-\frac{L}{2}\right)\right), & \text{for } \frac{L}{2} \le x \le L. \end{cases}$$

Force at the pile head:

$$\begin{aligned} F_h(0) &= -\left(EA\,\frac{du_h}{dx}\right)(0) \\ &\approx \left[1.11543(-2) + 0.60875(2)\right] P_0 \\ &\approx -1.01337\, P_0. \end{aligned}$$

(better than before with 1 element!)