Problem 1.

The potential energy is given by,

$$\Pi(\underline{u}) = \frac{1}{2}\Big[\frac{EA}{L}\big((u_{B,x}-u_{A,x})(0) + (u_{B,y}-u_{A,y})(1)\big)^2 + \frac{EA}{L}\big((\cancel{u_{C,x}} - u_{B,x})(1) + (\cancel{u_{C,y}} - u_{B,y})(0)\big)^2$$

$$+ \frac{EA}{L}\big((u_{D,x} - \cancel{u_{C,x}})(0) + (\overset{u_{Dx}}{\cancel{u_{D,y}}} - \cancel{u_{C,y}})(-1)\big)^2 + \frac{EA}{L}\big((u_{D,x} - u_{A,x})(1) + (\overset{u_{Dx}}{\cancel{u_{D,y}}} - u_{A,y})(0)\big)^2$$

$$+ \frac{EA}{L\sqrt{2}}\big((\cancel{u_{C,x}} - u_{A,x})(\tfrac{1}{\sqrt{2}}) + (\cancel{u_{C,y}} - u_{A,y})(\tfrac{1}{\sqrt{2}})\big)^2 \Big]$$

$$- (-P)\, u_{A,x}$$

$$= \frac{1}{2}\Big[\frac{EA}{L}(u_{B,y}-u_{A,y})^2 + \frac{EA}{L}(u_{B,x})^2 + \frac{EA}{L}(u_{D,x})^2 + \frac{EA}{L}(u_{D,x}-u_{A,x})^2 + \frac{EA}{L2\sqrt{2}}(u_{A,x}+u_{A,y})^2 \Big] + P u_{A,x}.$$

Differentiate with respect to $u_{A,x}$, $u_{A,y}$, $u_{B,x}$, $u_{B,y}$, u_{Dx} to find the gradient $\nabla\Pi$.

$$\frac{\partial \Pi}{\partial u_{A,x}} = \frac{EA}{L}(u_{D,x}-u_{A,x})(-1) + \frac{EA}{L2\sqrt{2}}(u_{A,x}+u_{A,y}) + P$$

$$\frac{\partial \Pi}{\partial u_{A,y}} = \frac{EA}{L}(u_{B,y}-u_{A,y})(-1) + \frac{EA}{L2\sqrt{2}}(u_{A,x}+u_{A,y})$$

$$\frac{\partial \Pi}{\partial u_{B,x}} = \frac{EA}{L}(u_{B,x})$$

$$\frac{\partial \Pi}{\partial u_{B,y}} = \frac{EA}{L}(u_{B,y}-u_{A,y})$$

$$\frac{\partial \Pi}{\partial u_{D,x}} = \frac{EA}{L}(u_{D,x}) + \frac{EA}{L}(u_{D,x}-u_{A,x}).$$

Set $\nabla\Pi = \underline{0}$ and we get,

$$\frac{EA}{L}\begin{bmatrix} 1+\frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & 0 & 0 & -1 \\ \frac{1}{2\sqrt{2}} & 1+\frac{1}{2\sqrt{2}} & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} u_{A,x} \\ u_{A,y} \\ u_{B,x} \\ u_{B,y} \\ u_{D,x} \end{bmatrix} = \begin{bmatrix} -P \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

This is the stiffness equation (reduced) that we had obtained before.

Problem 2.

We have,

$$\pi(\underline{u}) = \frac{1}{2}\Big[\frac{EA}{8L}\Big(\big(\overset{u_0}{\cancel{u_{O.x}}} - \cancel{u_{A.x}}\big)\Big(\frac{4\sqrt{3}}{8}\Big) + \big(u_{O.y} - \cancel{u_{A.y}}\big)\Big(-\frac{1}{2}\Big)\Big)^2 + \frac{EA}{4L}\Big(\big(\overset{u_0}{\cancel{u_{O.x}}} - \cancel{u_{B.x}}\big)(0) + \big(u_{O.y} - \cancel{u_{B.y}}\big)(-1)\Big)^2$$

$$+ \frac{EA}{5L}\Big(\big(\overset{u_0}{\cancel{u_{O.x}}} - \cancel{u_{C.x}}\big)\Big(-\frac{3}{5}\Big) + \big(u_{O.y} - \cancel{u_{C.y}}\big)\Big(-\frac{4}{5}\Big)\Big)^2\Big]$$

$$-(-P)\,u_{O.y}.$$

$$= \frac{1}{2}\Big[\frac{EA}{8L}\Big(\frac{4\sqrt{3}}{8}u_0 - \frac{1}{2}u_{O.y}\Big)^2 + \frac{EA}{4L}(u_{O.y})^2 + \frac{EA}{5L}\Big(\frac{3}{5}u_0 + \frac{4}{5}u_{O.y}\Big)^2\Big] + P\,u_{O.y}.$$

Differentiate π with respect to $u_{O.y}$:

$$\frac{\partial\pi}{\partial u_{O.y}} = \frac{EA}{8L}\Big(\frac{4\sqrt{3}}{8}u_0 - \frac{1}{2}u_{O.y}\Big)\Big(-\frac{1}{2}\Big) + \frac{EA}{4L}(u_{O.y}) + \frac{EA}{5L}\Big(\frac{3}{5}u_0 + \frac{4}{5}u_{O.y}\Big)\Big(\frac{4}{5}\Big) + P.$$

Set $\nabla\pi = 0$ and solve for $u_{O.y}$:

$$\frac{EA}{L}\underbrace{\Big(\frac{1}{32} + \frac{1}{4} + \frac{16}{125}\Big)}_{1637/4000}\,u_{O.y} = -P - \frac{EA}{L}\Big(-\frac{4\sqrt{3}}{128} + \frac{12}{125}\Big)u_0$$

$$\Rightarrow \boxed{u_{O.y} = -\Big(\frac{4000}{1637}\Big)\frac{PL}{EA} - \Big(\frac{4000}{1637}\Big)\Big(\frac{12}{125} - \frac{\sqrt{3}}{32}\Big)u_0}$$

Problem 3.

(a) Recall the governing equation for the axial displacement. With $EA = 1$, we just get,

$$-\frac{d^2u}{dx^2} = \frac{1}{2}(x-2), \quad \text{for} \quad 0<x<4.$$

Since the bar is fixed to the wall, we must have $u(0)=0$. It is subjected to a load of $P = \frac{5}{12}$ on the right end, so $(EAu')(4) = -P = -\frac{5}{12}$.

Hence, the strong form is,

Find $u \in C^2(0,4)$ such that

$$-\frac{d^2u}{dx^2} = \frac{1}{2}(x-2), \quad \text{for} \quad 0<x<4$$

Subject to the BCs

$$u(0)=0, \quad u'(4) = -\frac{5}{12}.$$

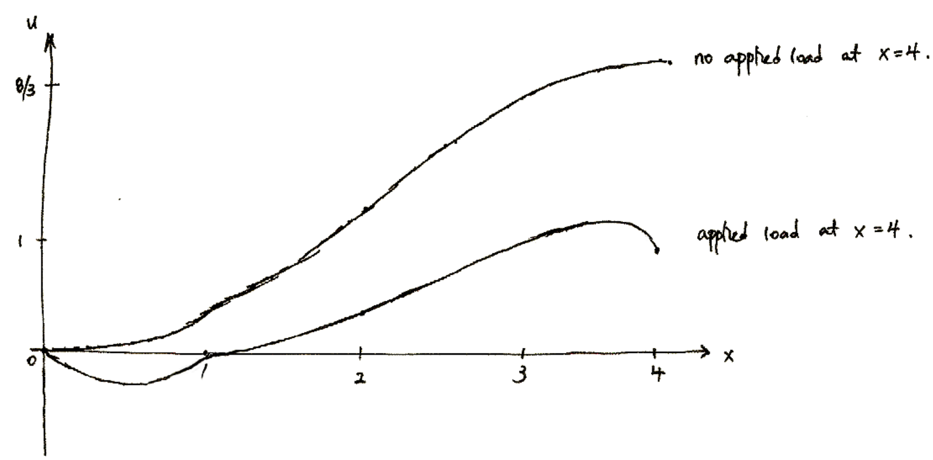
To find the exact solution, we integrate the differential equation twice. We get,

$$u'(x) = -\frac{1}{4}x^2 + x + c_0, \qquad u'(4) = -4+4+c_0 \Rightarrow c_0 = -\frac{5}{12}$$

$$u(x) = -\frac{1}{12}x^3 + \frac{1}{2}x^2 - \frac{5}{12}x + c_1, \qquad u(0) = c_1 \Rightarrow c_1 = 0.$$

Hence, the exact solution is,

$$\boxed{u(x) = -\frac{1}{12}x^3 + \frac{1}{2}x^2 - \frac{5}{12}x, \quad \text{for} \quad 0 \le x \le 4.}$$



(b) To find the weak form, we multiply the equation by an admissible test function φ, and integrate by parts:

$$-\int_0^4 u''\varphi \, dx = \int_0^4 \tfrac{1}{2}(x-2)\varphi \, dx$$

$$\Rightarrow \int_0^4 u'\varphi' \, dx - [u'\varphi]_0^4 = \int_0^4 \tfrac{1}{2}(x-2)\varphi \, dx$$

$$\Rightarrow \int_0^4 u'\varphi' \, dx = \int_0^4 \tfrac{1}{2}(x-2)\varphi \, dx + \underbrace{u'(4)}_{=-\frac{5}{12}}\varphi(4) - u'(0)\underbrace{\varphi(0)}_{\equiv 0}$$

$\equiv 0$ (since $u'(0)$ is unknown; equivalent to eliminating a row to get the reduced stiffness matrix)

$$\Rightarrow \int_0^4 u'\varphi' \, dx = \int_0^4 \tfrac{1}{2}(x-2)\varphi \, dx - \tfrac{5}{12}\varphi(4).$$

Weak form:

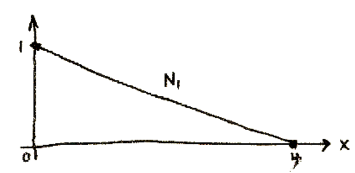
Let $V = \{$ all functions u from $H^1(0,4)$ space that satisfy $u(0)=0 \}$.
Find $u \in V$ such that,

$$\int_0^4 u'\varphi' \, dx = \int_0^4 \tfrac{1}{2}(x-2)\varphi \, dx - \tfrac{5}{12}\varphi(4), \quad \forall \varphi \in V.$$

Let $S = \{$ all piecewise linear functions u_h that satisfy $u_h(0) = 0 \}$.
Find $u_h \in S$ such that

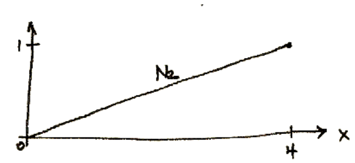
$$\int_0^4 u_h'\varphi_h' \, dx = \int_0^4 \tfrac{1}{2}(x-2)\varphi_h \, dx - \tfrac{5}{12}\varphi_h(4), \quad \forall \varphi_h \in S.$$

(c). 1 element. $S = \text{span}\{N_1, N_2\}$, where



$N_1(x) = -\frac{1}{4}(x-4), \quad 0 \le x \le 4$

$N_1'(x) = -\frac{1}{4}, \quad 0 < x < 4$



$N_2(x) = \frac{1}{4}x, \quad 0 \le x \le 4$

$N_2'(x) = \frac{1}{4}, \quad 0 < x < 4.$

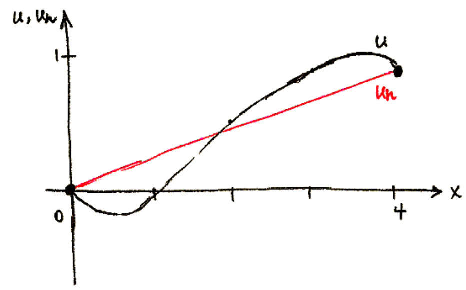
$$K = \begin{bmatrix} \int_0^4 N_1'N_1' \, dx & \int_0^4 N_2'N_1' \, dx \\ \int_0^4 N_1'N_2' \, dx & \int_0^4 N_2'N_2' \, dx \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{1}{4} \end{bmatrix}, \quad f = \underbrace{\begin{bmatrix} \int_0^4 \frac{1}{2}(x-2)N_1 \, dx \\ \int_0^4 \frac{1}{2}(x-2)N_2 \, dx \end{bmatrix}}_{(-2/3,\ 2/3)} + \begin{bmatrix} 0 \\ -\frac{5}{12} \end{bmatrix} = \begin{bmatrix} -\frac{2}{3} \\ \frac{1}{4} \end{bmatrix}.$$

Apply $u_1 = 0$, and solve for the other component:

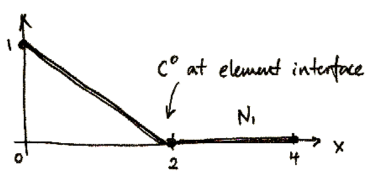
$$[u_2] = \left[\tfrac{1}{4}\right]^{-1}\left[\tfrac{1}{4}\right] = [1].$$

Hence,

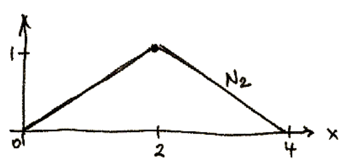
$$u_h(x) = 0 \cdot N_1(x) + 1 \cdot N_2(x) = \boxed{\tfrac{1}{4}x, \quad 0 \le x \le 4}$$



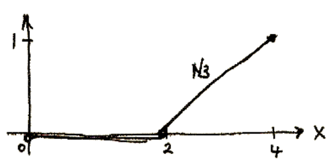
<u>2 elements</u>. $S = \text{span}\{N_1, N_2, N_3\}$, where



$$N_1(x) = \begin{cases} -\frac{1}{2}(x-2), & 0 \le x \le 2 \\ 0, & 2 \le x \le 4. \end{cases} \qquad N_1'(x) = \begin{cases} -\frac{1}{2}, & 0 < x < 2 \\ 0, & 2 < x < 4 \end{cases}$$



$$N_2(x) = \begin{cases} \frac{1}{2}x, & 0 \le x \le 2 \\ -\frac{1}{2}(x-4) & 2 \le x \le 4 \end{cases} \qquad N_2'(x) = \begin{cases} \frac{1}{2}, & 0 < x < 2 \\ -\frac{1}{2}, & 2 < x < 4 \end{cases}$$



$$N_3(x) = \begin{cases} 0, & 0 \le x \le 2 \\ \frac{1}{2}(x-2), & 2 \le x \le 4 \end{cases} \qquad N_3'(x) = \begin{cases} 0, & 0 < x < 2 \\ \frac{1}{2}, & 2 < x < 4. \end{cases}$$

Because the derivatives of N_1, N_2, N_3 are discontinuous at the element interfaces (but are continuous otherwise), we must evaluate the stiffness entries "element-wise," e.g.

$$K_{11} = \int_0^4 N_1' N_1' \, dx = \int_0^2 \left(-\tfrac{1}{2}\right)\left(-\tfrac{1}{2}\right) dx + \int_2^4 (0)(0)\, dx = \tfrac{1}{2}.$$

$$K_{12} = \int_0^4 N_2' N_1' \, dx = \int_0^2 \left(\tfrac{1}{2}\right)\left(-\tfrac{1}{2}\right) dx + \int_2^4 \left(-\tfrac{1}{2}\right)(0)\, dx = -\tfrac{1}{2}$$

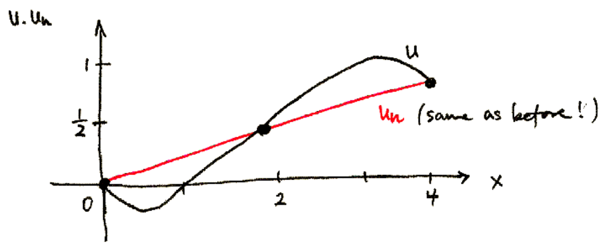
$$K_{13} = \int_0^4 N_3' N_1' \, dx = \underbrace{\int_0^2 (0)\left(-\tfrac{1}{2}\right) dx}_{\text{element ①}} + \underbrace{\int_2^4 \left(\tfrac{1}{2}\right)(0)\, dx}_{\text{element ②}} = 0, \qquad \text{etc.}$$

We find that,

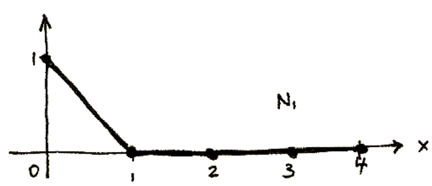
$$K = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \\ -\frac{1}{2} & \frac{1}{2}+\frac{1}{2} & -\frac{1}{2} \\ & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}, \qquad f = \begin{bmatrix} -\frac{2}{3} \\ -\frac{1}{3}+\frac{1}{3} \\ +\frac{2}{3} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -\frac{5}{12} \end{bmatrix} = \begin{bmatrix} -\frac{2}{3} \\ 0 \\ \frac{1}{4} \end{bmatrix}$$

set $u_1 = 0$:

$$\begin{bmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{4} \end{bmatrix} \Rightarrow \begin{bmatrix} u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}. \qquad u_h(x) = \frac{1}{2}N_2(x) + 1\cdot N_3(x)$$



__4 elements.__ $S = \text{span}\{N_1, N_2, N_3, N_4, N_5\}$,

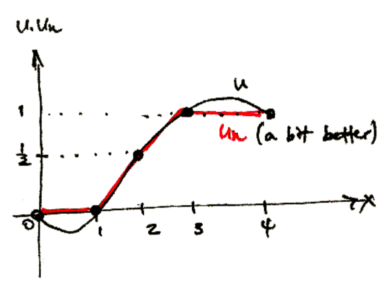


$$N_1(x) = \begin{cases} -(x-1), & 0 \le x \le 1 \\ 0, & 1 \le x \le 2 \\ 0, & 2 \le x \le 3 \\ 0, & 3 \le x \le 4 \end{cases} \qquad N_1'(x) = \begin{cases} -1, & 0 < x < 1 \\ 0, & 1 < x < 2 \\ 0, & 2 < x < 3 \\ 0, & 3 < x < 4 \end{cases}.$$

etc.

$$K = \begin{bmatrix} 1 & -1 & & & \\ -1 & 1+1 & -1 & & \\ & -1 & 1+1 & -1 & \\ & & -1 & 1+1 & -1 \\ & & & -1 & 1 \end{bmatrix}, \qquad f = \begin{bmatrix} -\frac{5}{12} \\ -\frac{1}{3} - \frac{1}{6} \\ -\frac{1}{12} + \frac{1}{12} \\ \frac{1}{6} + \frac{1}{3} \\ \frac{5}{12} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -\frac{5}{12} \end{bmatrix} = \begin{bmatrix} -\frac{5}{12} \\ -\frac{1}{2} \\ 0 \\ \frac{1}{2} \\ 0 \end{bmatrix}.$$

$$\begin{bmatrix} u_2 \\ u_3 \\ u_4 \\ u_5 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{2} \\ 1 \\ 1 \end{bmatrix}. \qquad u_h(x) = \frac{1}{2}N_3(x) + N_4(x) + N_5(x)$$



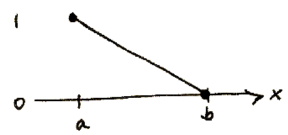
Element assembly

From the cases of 2 and 4 elements, we saw that the integral over a union of elements is the sum of the integrals restricted to an element, e.g. $(0,4) = e_1 \cup e_2$, and,

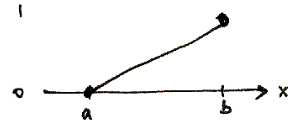
$$\int_0^4 N_i' N_i' \, dx = \underbrace{\int_0^2 N_i' N_i' \, dx}_{(e_1)} + \underbrace{\int_2^4 N_i' N_i' \, dx}_{(e_2)} .$$

We note that the piecewise linear basis functions, when restricted to an element, always take one of the following forms ("shapes"):

(1)



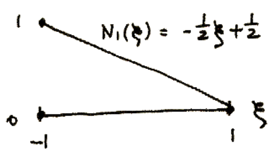
(2)

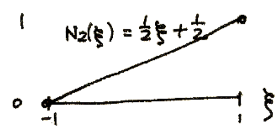


(3)

(not too interesting, as this gives zero contribution to both the stiffness matrix and the RHS vector)

So it is useful to consider what happens for the stiffness matrix and the RHS vector in the so-called parent (or parametric) domain. We will use $[-1,1]$ in anticipation of numerical integration techniques:



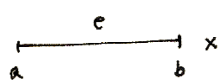


(note the abuse of notation; here, N_1, N_2 denote the basis function in the parent domain)

Once we compute the entries, we can appropriately map to the physical domain by coordinate transformation: (assumed here to be linear.)

$$\xrightarrow{\quad} \qquad x = \frac{b-a}{2}(1+\xi) + a$$

$$dx = \frac{b-a}{2} \, d\xi$$



Note that the coordinate map $x = x(\xi)$ is invertible (i.e. nonzero/positive Jacobian) at all points, and the inverse map is given by,

$$\xleftarrow{\quad} \qquad \xi = \frac{2}{b-a}(x-a) - 1$$

$$d\xi = \frac{2}{b-a} \, dx .$$

Now, the element stiffness matrix in the parent domain is given by,

$$K^e(\xi) = \int_{-1}^{1} \begin{bmatrix} \frac{dN_1}{d\xi}\frac{dN_1}{d\xi} & \frac{dN_2}{d\xi}\frac{dN_1}{d\xi} \\ \frac{dN_1}{d\xi}\frac{dN_2}{d\xi} & \frac{dN_2}{d\xi}\frac{dN_2}{d\xi} \end{bmatrix} d\xi = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

(integral of matrix = matrix of integrals)

Hence, the element stiffness matrix in the physical domain (a, b) is given by,

$$K^e(x) = \int_a^b \begin{bmatrix} \frac{dN_1}{dx}\frac{dN_1}{dx} & \frac{dN_2}{dx}\frac{dN_1}{dx} \\ \frac{dN_1}{dx}\frac{dN_2}{dx} & \frac{dN_2}{dx}\frac{dN_2}{dx} \end{bmatrix} dx \qquad \frac{dN_1}{dx} = \left(\frac{dN_1}{d\xi}\right)\left(\frac{d\xi}{dx}\right) = \left(\frac{dN_1}{d\xi}\right)\left(\frac{2}{b-a}\right), \text{ etc.}$$

$$= \int_{-1}^{1} \begin{bmatrix} \left(\frac{dN_1}{d\xi}\cdot\frac{2}{b-a}\right)\left(\frac{dN_1}{d\xi}\cdot\frac{2}{b-a}\right) & \left(\frac{dN_2}{d\xi}\cdot\frac{2}{b-a}\right)\left(\frac{dN_1}{d\xi}\cdot\frac{2}{b-a}\right) \\ \left(\frac{dN_1}{d\xi}\cdot\frac{2}{b-a}\right)\left(\frac{dN_2}{d\xi}\cdot\frac{2}{b-a}\right) & \left(\frac{dN_2}{d\xi}\cdot\frac{2}{b-a}\right)\left(\frac{dN_2}{d\xi}\cdot\frac{2}{b-a}\right) \end{bmatrix} \left(\frac{b-a}{2}\, d\xi\right)$$

$$= \underbrace{\frac{2}{b-a}\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}}_{\text{you may recognize this from } \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.}.$$

As for the force vector (for the distributed load),

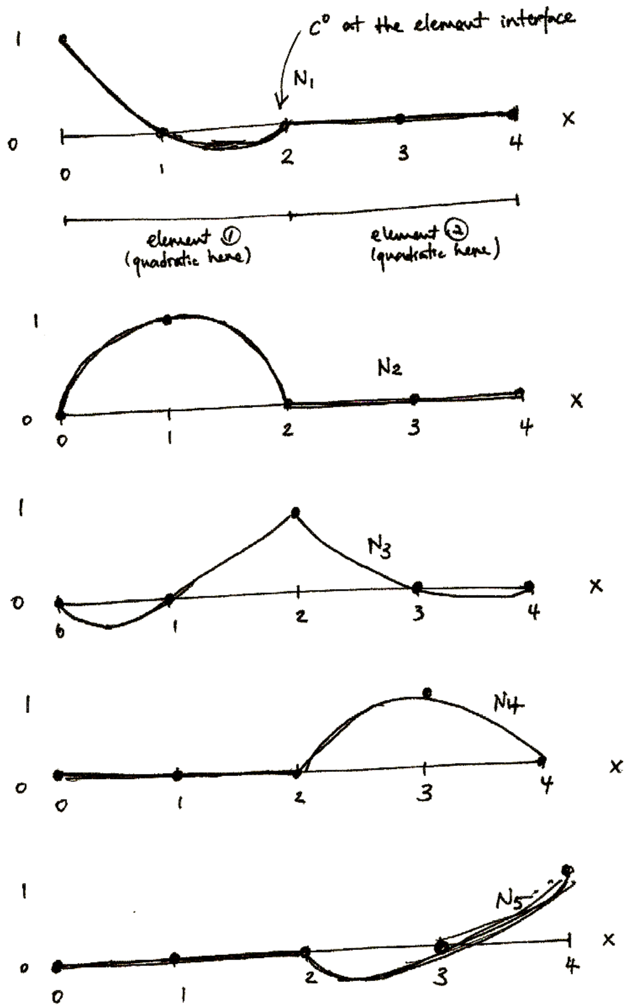
$$\underbrace{f^e(x)}_{\text{element force vector}} = \int_a^b \begin{bmatrix} \frac{1}{2}(x-2)\, N_1(x) \\ \frac{1}{2}(x-2)\, N_2(x) \end{bmatrix} dx \qquad x = \frac{b-a}{2}(1+\xi) + a$$

$$= \int_{-1}^{1} \underbrace{\begin{bmatrix} \frac{1}{2}\left[\left(\frac{b-a}{2}(1+\xi) - a\right) - 2\right]\left(-\frac{1}{2}\xi + \frac{1}{2}\right) \\ \frac{1}{2}\left[\left(\frac{b-a}{2}(1+\xi) - a\right) - 2\right]\left(\frac{1}{2}\xi + \frac{1}{2}\right) \end{bmatrix}}_{\substack{\text{cumbersome to integrate!} \\ \text{(need numerical methods of integration)}}} \left(\frac{b-a}{2}\, d\xi\right)$$

Piecewise quadratic basis functions

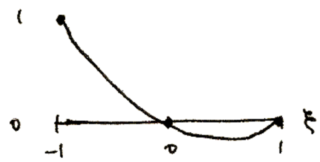
These functions are still only C^0-continuous throughout the entire domain (i.e. does not provide us additional smoothness), but do provide some flexibility in modeling the fields in question.

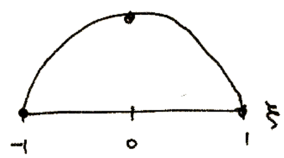
In the physical domain (say, for 2 elements), the basis functions look like these:

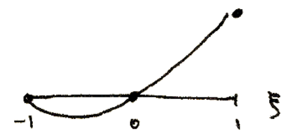


$$u(x) \approx u_h(x) \equiv u_1 N_1(x) + u_2 N_2(x) + u_3 N_3(x) + u_4 N_4(x) + u_5 N_5(x)$$

In the parent domain (i.e. restricted to an element and mapped to $[-1,1]$), we have







$N_1(\xi) = \frac{1}{2}\xi^2 - \frac{1}{2}\xi$ $\qquad$ $N_2(\xi) = -\xi^2+1$ $\qquad$ $N_3(\xi) = \frac{1}{2}\xi^2+\frac{1}{2}\xi$.

To easily determine these functions, we can consider the method of <u>Lagrange interpolation</u>. For example, $N_1(\xi)$ is supposed to go through (interpolate) the points $(0,0)$ and $(1,0)$, so we know that N_1 takes the form,

$$N_1(\xi) = c(\xi-0)(\xi-1),$$

for some constant c. This constant is determined by the remaining point $(-1,1)$.

$$1 = c(-1-0)(-1-1) \Rightarrow c = \frac{1}{2}.$$

Hence,

$$N_1(\xi) = \frac{1}{2}\xi(\xi-1) = \frac{1}{2}\xi^2 - \frac{1}{2}\xi.$$

etc.

The element stiffness matrix in the parent domain is given by,

$$K^e(\xi) = \int_{-1}^{1} \begin{bmatrix} \frac{dN_1}{d\xi}\frac{dN_1}{d\xi} & \frac{dN_2}{d\xi}\frac{dN_1}{d\xi} & \frac{dN_3}{d\xi}\frac{dN_1}{d\xi} \\ \frac{dN_1}{d\xi}\frac{dN_2}{d\xi} & \frac{dN_2}{d\xi}\frac{dN_2}{d\xi} & \frac{dN_3}{d\xi}\frac{dN_2}{d\xi} \\ \frac{dN_1}{d\xi}\frac{dN_3}{d\xi} & \frac{dN_2}{d\xi}\frac{dN_3}{d\xi} & \frac{dN_3}{d\xi}\frac{dN_3}{d\xi} \end{bmatrix} d\xi = \begin{bmatrix} \frac{7}{6} & -\frac{4}{3} & \frac{1}{6} \\ -\frac{4}{3} & \frac{8}{3} & -\frac{4}{3} \\ \frac{1}{6} & -\frac{4}{3} & \frac{7}{6} \end{bmatrix}$$

$$\Rightarrow \quad K^e(x) = \frac{2}{b-a}\begin{bmatrix} \frac{7}{6} & -\frac{4}{3} & \frac{1}{6} \\ -\frac{4}{3} & \frac{8}{3} & -\frac{4}{3} \\ \frac{1}{6} & -\frac{4}{3} & \frac{7}{6} \end{bmatrix}.$$

<u>1 element.</u>

$$K = \frac{2}{4}\begin{bmatrix} \frac{7}{6} & -\frac{4}{3} & \frac{1}{6} \\ -\frac{4}{3} & \frac{8}{3} & -\frac{4}{3} \\ \frac{1}{6} & -\frac{4}{3} & \frac{7}{6} \end{bmatrix}, \qquad f = \underbrace{\begin{bmatrix} -\frac{2}{3} \\ 0 \\ \frac{2}{3} \end{bmatrix}}_{\text{due to distributed load } f(x)} + \begin{bmatrix} 0 \\ 0 \\ -\frac{5}{12} \end{bmatrix} = \begin{bmatrix} -\frac{2}{3} \\ 0 \\ \frac{1}{4} \end{bmatrix}$$

Apply $u_1 = 0$ and solve for u_2, u_3 :

$$\frac{1}{2}\begin{bmatrix} \frac{8}{3} & -\frac{4}{3} \\ -\frac{4}{3} & \frac{7}{6} \end{bmatrix}\begin{bmatrix} u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{4} \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}.$$

Hence,

$$u_h(x) = 0 \cdot N_1(x) + \frac{1}{2} \cdot N_2(x) + 1 \cdot N_3(x) = \frac{1}{4}x \quad \text{(degenerated to a linear function!)}$$

<u>2 elements</u>

$$K = \frac{2}{2}\begin{bmatrix} \frac{7}{6} & -\frac{4}{3} & \frac{1}{6} & & \\ -\frac{4}{3} & \frac{8}{3} & -\frac{4}{3} & & \\ \frac{1}{6} & -\frac{4}{3} & \frac{7}{6}+\frac{7}{6} & -\frac{4}{3} & \frac{1}{6} \\ & & -\frac{4}{3} & \frac{8}{3} & -\frac{4}{3} \\ & & \frac{1}{6} & -\frac{4}{3} & \frac{7}{6} \end{bmatrix}, \qquad f = \begin{bmatrix} -\frac{1}{3} \\ -\frac{2}{3} \\ 0 + 0 \\ \frac{2}{3} \\ \frac{1}{3} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -\frac{5}{12} \end{bmatrix} = \begin{bmatrix} -\frac{1}{3} \\ -\frac{2}{3} \\ 0 \\ \frac{2}{3} \\ -\frac{1}{12} \end{bmatrix}$$

Set $u_1 = 0$. Then,

$$u_h(x) = 0 \cdot N_1(x) + 0 \cdot N_2(x) + \frac{1}{2} \cdot N_3(x) + 1 \cdot N_4(x) + 1 \cdot N_5(x)$$
$$= \begin{cases} \frac{1}{4}x^2 - \frac{1}{4}x, & 0 \le x \le 2 \\ -\frac{1}{4}x^2 + \frac{7}{4}x - 2, & 2 \le x \le 4. \end{cases}$$